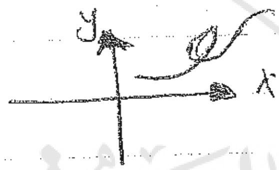
Ch 13 Vector-Valued Functions and motion in space.

13.1 Curves in space and Their tangents.

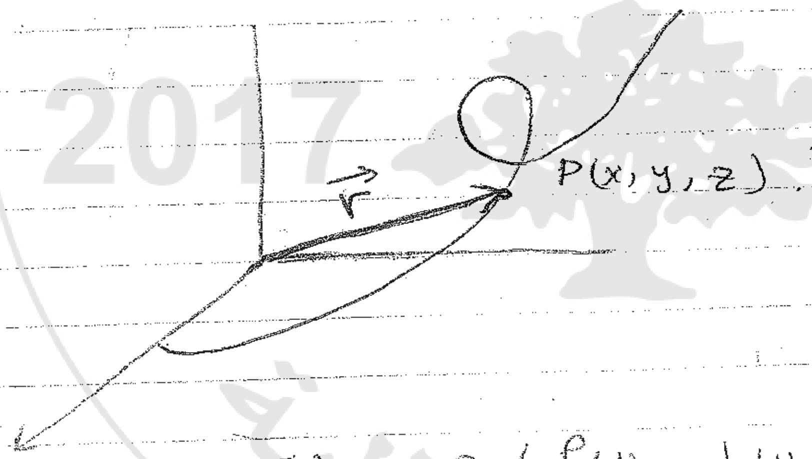


* parametrization $\boxed{11.1}$

يعني عندي احداثيات تتحرك باتجاه xy ويكون اتجاه الحركة
يعتمد على الزمن.

$x, y \longrightarrow t$.

$(t) \Rightarrow$ parameter. $t \in I$. يعني الزمن (t).



$P(x,y,z) = P\,(f(t),\ h(t),\ g(t))$.

= Curve in space

= path.

$\vec{r}(t)$ = position vector

$= (f(t))\hat{i} + (h(t))\hat{j} + (g(t))\ k$.

$t \in R$.

$\vec{r}(t) : I \longrightarrow$ vector in space.

$f(t)\ ,\ h(t)\ ,\ g(t) \equiv$ component $\equiv$ scalars.

$\vec{r}(t)$ = vector - valued function
= vector - function.

Def :- A vector - valued function on a domain D is a rule that assigns a vector for each element in D.

Note : A real - valued functions is called a scaler function

$$\vec{r}(t) = \underbrace{f(t)}\,\hat{i} + \underbrace{g(t)}\,\hat{j} + \underbrace{h(t)}\,k.$$

scalars.

Ex] $\vec{r}(t) = (\sin t)\,i + (\tan t)\,\hat{j} + (\sqrt{1-t^2})\,k.$

$\vec{r}(0) = 0 + 0 + k.$

$t \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ -

$1 - t^2 \geq 0$

D of $\vec{r}(t)$: R

$t^2 \leq 1$

$|t| \leq 1$

ملا ال (D) هو عبارة عن تقاطع كل الاقترانات

$D\ \vec{r}(t)$: $\underline{Df} \cap \underline{Dg} \cap \underline{Dh}.$

Ex) $\vec{r}(t) = (\cos t)i + (\sin t)j + tK.$

Domain : $(-\infty, \infty)$.

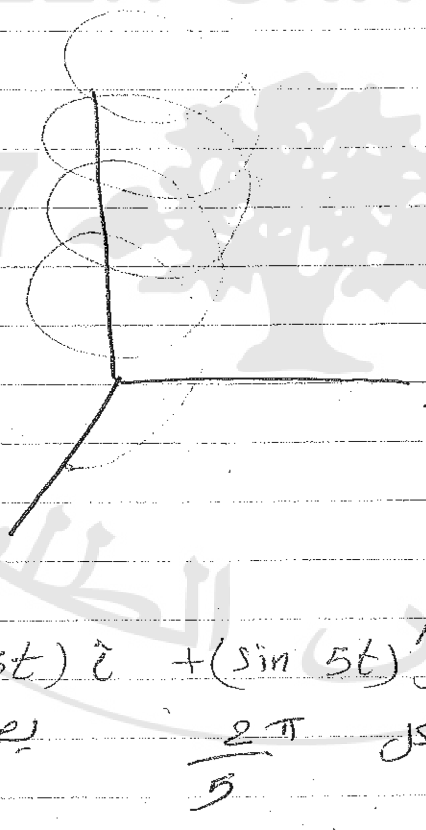
$\vec{r}(0) = i$

$\vec{r}(1) = \cos(1)\, i + \sin 1\, j + K.$

$r(\pi/2) = (\cos \pi/2)\, i + (\sin \pi/2)j + \pi/2\, K.$

$= \hat{j} + \pi/2\, K.$

$r(\pi) = -\hat{i} + \pi K.$



يسمى spiral

أو Helix.

$\cos^2 t + \sin^2 t = 1$

circle.

يكرر نفسه كل 2π.

$\vec{r}(t) = (\cos 5t)\hat{i} + (\sin 5t)\hat{j} + tK.$

strich. يصيبه   يكرر نفسه كل $\frac{2\pi}{5}$

Def :- If $\vec{r}(t) = f(t)\,i + g(t)\,\hat{j} + h(t)\,k$ is a vector function, then

$$\lim_{t \to t_0} \vec{r}(t) = \lim_{t \to} \left[f(t)\,i + g(t)\,j + h(t)\,k \right]$$

$$= \lim_{t \to t_0} f(t)\,\hat{i} + \lim_{t \to t_0} g(t)\,j + \lim_{t \to t_0} h(t)\,k$$

$$= L_1\,\hat{i} + L_2\,\hat{j} + L_3\,k$$

* $\lim \vec{r}(t) = \vec{L} \iff \left.\begin{matrix} \lim f(t) \\ \lim g(t) \\ \lim h(t) \end{matrix}\right\}$ exist

Ex
$$\vec{r}(t) = (\sin t)\,i + \tan t\,\hat{j} + e^{t^2+1}\,k$$

~~lim $\vec{r}(t)$~~ $\lim \vec{r}(0) = 0 + 0 + e^{1} k = e\,k$

$$\lim_{t \to \pi/4} \vec{r}(t) = \underbrace{\frac{1}{\sqrt{2}}}_{L_1}\, i + \underbrace{\hat{j}}_{L_2} + \underbrace{\left(e^{\left(\frac{\pi}{4}\right)^2+1} \right)}_{L_3} k$$

Vector Valued Functions :-

$\vec{v}(t) = f(t)\,i + g(t)\,j + h(t)\,k, \qquad t \in I.$

$\lim_{t \to t_0} \vec{v}(t) = \vec{L} = L_1 i + L_2 j + L_3 k.$

Continuity :-

$\vec{v}(t)$ is cont at $t = t_0$. if :-

1) $\lim_{t \to t_0} \vec{v}(t)$ exists

2) $\vec{v}(t_0)$ is definet.

3) $\lim \vec{v}(t) = \vec{v}(t_0)$

* $v(t)$ is continuous if it is cont. at each point of its domain.

Ex/ :-

1) $\vec{v}(t) = (e^{-2t})\,i + (\cos t)\,\hat{j} + t^2\,k.$

cont for all of t.

2) $\vec{v}(t) = t^2 i + t j + \lfloor t \rfloor k.$

$\lfloor t \rfloor$ is cont for all t not integer.

$r(t)$ cont for t not integer.

Derivative of vector-valued functions.

suppose that $\vec{r}(t) = f(t)\,i + g(t)\,j + h(t)\,k$.

suppose that $f(t)$, $g(t)$, and $h(t)$ are diff. function of t

if $t \to$ change from t to $t+\Delta t$.

then $$\frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$= \left[\frac{f(t+\Delta t)\hat{i} + g(t+\Delta t)\hat{j} + h(t+\Delta t)\,k}{\Delta t} \right]$$

$$= \left(\frac{f(t+\Delta t) - f(t)}{\Delta t} \right) i + \cdots + \cdots$$

provided that limit exist.

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}\, i + \cdots + \cdots$$

$$= f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\,k.$$

$\Rightarrow$ If $\vec{r}(t) = f(t)\,i + g(t)\,\hat{j} + h(t)\,K$

then $\vec{r}'(t) = f'(t)\,\hat{i} + g'(t)\,\hat{j} + h'(t)\,K$.

$\vec{r}(t)$ is diff $\Longleftrightarrow$ $f(t)$, $g(t)$ and $h(t)$ are all diff.

13.2 :- Integrals of Vector Functions.

Def: ~~All~~ A differentiable vector function $\vec{R}(t)$ is an antiderivative of the vector function $\vec{r}(t)$ in an interval I if $\frac{d\vec{R}}{dt} = r(t)$.

The set of all antiderivatives is called the indefinite integral of $\vec{r}(t)$ with respect to t.

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$$

$\vec{C}$ = constant vector.

If $\vec{r}(t) = (f(t))\,i + (g(t))\,j + (h(t))\,k$

then:-

$$\int \vec{r}(t)\,dt = \int \left(f(t)\,dt\right) i \quad \int\left(g(t)\,dt\right) \hat{j} + \left(\int h(t)\,dt\right) k.$$

If $\vec{R}(t)$ is an antiderevative of $\vec{v}(t)$, then.

$$\int_a^b \vec{v}(t)\,dt = R(t)\Big]_a^b = \vec{R}(b) - \vec{R}(a)$$

Ex:- $\int_0^{\pi/3} \left[(\sec t \ \tan t)\,\hat{i} + \left((\cos t)\,\hat{j} + (2 \sin t \cos t)\,k\right) \right] dt.$

$$= (\sec t)\,\hat{i}\Big]_0^{\pi/3} + (\sin t\,\hat{j})\Big]_0^{\pi/3} + \left(\frac{\cos 2t}{2}\right) k\Big]_0^{\pi/3}$$

$$(2 - 1)\,\hat{i} + \frac{\sqrt{3}}{2}\,\hat{j} - \left(\frac{-1}{4} - \frac{1}{2}\right) k.$$

$$= \hat{i} + \frac{\sqrt{3}}{2}\,\hat{j} + \frac{3}{4}\,k$$

Ex] If $\vec{a}(t) = (-3\cos t)\hat{i} - (3\sin t)\hat{j} + 2\hat{k}$.

$v(0) = 3\hat{j}$, $\vec{r}(0) = 3\hat{i}$.

find $\vec{r}(t)$.

$\int a(t)\,dt = \vec{v}(t) = (-3\sin t)\hat{i} + (3\cos t)\hat{j} + 2t\hat{k} + \vec{c}$.

$\vec{v}(0) = 0 + 3\hat{j} + (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) = 3\hat{j}$

$(3 + c_2)\hat{j} = 3\hat{j} \rightarrow c_2 = 0$.

$\boxed{c = 0}$

$r(t) = \int \vec{v}(t)\,dt$.

$\boxed{13.3}$:- Arc Length and tangent vector

$y = f(x) \Rightarrow L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$.

$a \le x \le b$.

$x = g(y) \Rightarrow L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$.

$c \le y \le d$.

$x = x(t) = f(t)$
$y = y(t) = g(t)$. $\Rightarrow L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.
$a \le t \le b$.

suppose that $\vec{r}(t) = (x(t))\hat{i} + (y(t))\hat{j} + (z(t))\hat{k}$.
is smooth curve $a \le t \le b$, then the arc length of $\vec{r}(t)$ is given by.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

$$\frac{dr}{dt} = \vec{r}'(t) = \left(\frac{dx}{dt}\right) \hat{i} + \left(\frac{dy}{dt}\right) \hat{j} + \left(\frac{dz}{dt}\right) k.$$

$$|\vec{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$L = \int_a^b |v(t)|\, dt$$

Helex:

$\boxed{Ex}$:- $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t k$ $\quad 0 \le t \le \pi$

$$v(t) = \frac{dr}{dt} = (-\sin t)\, i + (\cos t)\, j + k$$

$$|\vec{v}| = \sqrt{2}$$

$$L = \int_0^{\pi} \sqrt{2}\, dt = \sqrt{2}\, \pi.$$

Ex] $\vec{r}(t) = e^t \cos t\, \hat{i} + (e^t \sin t)\hat{j} + e^t K$ $\quad -\ln 4 \le t \le 0$

$e^t - \sin t$

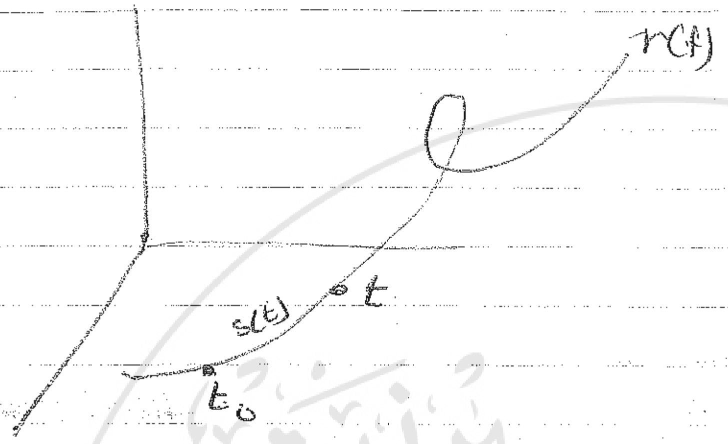
$$\vec{v}(t) = (e^t \cos t - e^t \sin t)\hat{i} + (e^t \sin t + e^t \cos t)\hat{j} + e^t k.$$

$$\vec{v}(t) = \sqrt{e^{2t}\left(\left[\cos^2 t + \sin^2 t - 2\sin t \cos t\right] + \sin^2 t + \cos^2 t + 2\sin t \cos t \right.}$$

$$= \sqrt{3e^{2t}} = \sqrt{3}\, e^t.$$

$$L = \int_{-\ln 4}^{0} \sqrt{3}\, e^t\, dt = \sqrt{3}\left[1 - e^{\ln -4}\right]. = (\sqrt{3})(1 - 1/4) = \frac{3\sqrt{3}}{4}$$

:- Arc length Parameter:



t_o = Initial position. (Initial point.).
Base point.

$$s(t) = \int_{t_o}^{t} |\vec{v}(z)|\, dz$$
z ↓ taw

s : is call the arc length parameter.

$$s'(t) = \frac{ds}{dt} = |\vec{v}(t)|$$

$s'(t) = \frac{ds}{dt} = |V(t)| \Rightarrow S.$

Ex) $r(t) = \cos(t)\,\hat{i} + (\sin t)\,\hat{j} + t\,k$

$|\vec{v}(t)| = \sqrt{2}.$

$$s(t) = \int_0^t \sqrt{2}\, dz = \sqrt{2}\, t.$$

$s = \int \sqrt{2}\, dt$

$s = \sqrt{2}\, t$

$t = \frac{s}{\sqrt{2}}$

$$s(t) = \sqrt{2}\, t \Rightarrow t = \frac{s}{\sqrt{2}}$$

$$\vec{r}(t) = r(t(s)) = \left(\cos\frac{s}{\sqrt{2}}\right)i + \left(\sin\frac{s}{\sqrt{2}}\right)j + \frac{s}{\sqrt{2}}\, k.$$

~~$S(t) = f(t)$~~

$\boxed{\# 13}$ $\vec{r}(t) = (\sqrt{2}\,t)\hat{i} + (\sqrt{2}\,t)\hat{j} + (1-t^2)K.$

8/ $(0,0,1)$ to $(\sqrt{2}, \sqrt{2}, 0).$

نقطة الأولى | $\sqrt{2}t = 0$ $\hat{i}$ | $t_0 = 0$ | $\sqrt{2}t = 0$ $\hat{j}$ | $t = 0$ | $1 - t^2 = 1$ K. | $t = 0.$

$\therefore$ $\boxed{t_0 = 0}$ to $\boxed{t = 1}$ من النقطة الثانية

$$\int |v(t)|\, dt = \int_0^1 2\sqrt{1+t^2}\, dt.$$

نفرض $\tan t = t^2$

$$\left|\int^{(t)} |v(\tau)|\, d\tau\right| =$$

$$\int_{t_0}^{t_1} |v(t)\, dt|$$

Unit Tangent vector $\vec{T}$:-

⇒ $r(t)$ = position vector (smooth curve).

⇒ $\vec{v}(t) = \dfrac{d\vec{r}}{dt}$.

Let $\vec{T} = \dfrac{\vec{v}}{|\vec{r}|}$ → T is called a unit tangent vector

(Direction of motion

Ex:- $\vec{r}(t) = (12\cos 2t)\hat{i} - (12\sin 2t)\hat{j} + 5t\hat{k}$

Find $\vec{T}$

$$\vec{r} = \frac{d\vec{r}}{dt} = (-24\sin 2t)\hat{i} - (24\cos 2t)\hat{j} + 5k.$$

$$|\vec{v}| = \sqrt{(24)^2[\sin^2 2t + \cos^2 2t] + 25}$$

$$= \sqrt{(24)^2 + 25} = \sqrt{601}$$

$$T = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{601}}\left[-(24\sin 2t)i - (24\cos 2t)\hat{j} + 5\hat{k}\right]$$

$\boxed{13.4}$:- Curvature and normal vector of curve :-

$\Rightarrow \vec{r}(t) =$ position vector.

$\Rightarrow \vec{v}(t) \Rightarrow T(t)$.

T has a constant magnitued it changes direction

$$s(t) = \int_{t_0}^{t} |v(z)| \, dz.$$

$y \xrightarrow{f(u)} u \xrightarrow{g(x)}$

$\Rightarrow$ Chain rule:-

$y = f(u)$.

$u = g(x)$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Def :- The rate at which the unit vector $\vec{T}$ turnes per unit of length along a smooth curve is called the curvature and it is denoted by K (kappa).

$$K = \left| \frac{d\vec{T}}{ds} \right| = \left| \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds} \right| = \frac{\left| \frac{d\vec{T}}{dt} \right|}{\left| \frac{ds}{dt} \right|} = \frac{\left| \frac{d\vec{T}}{dt} \right|}{\vec{v}} = \frac{1}{|\vec{v}|} \cdot \left| \frac{d\vec{T}}{dt} \right|$$

$\vec{r}(t) \rightarrow \vec{v} \rightarrow |\vec{v}| \rightarrow \vec{T} \rightarrow \frac{dT}{dt}$ يعني اشكال $\Leftarrow$

$|\vec{v}| \neq$ zero.

Ex :- $r(t) = \vec{r_0} + t\vec{u}$. find K for the line.

$\vec{r_0}$ = constant vector.

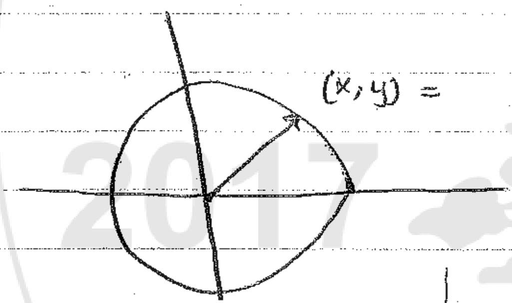
$\vec{u}$ = " "

$\vec{v} = \frac{\vec{dr}}{dt} = \vec{u}$,

$T = \cancel{\quad} = \frac{\vec{v}}{|\vec{v}|} = \frac{\text{constant}}{\text{const}} = \text{con}$

$\frac{dT}{dt} = 0 \rightarrow K = 0$.

Ex :- circle.



$x^2 + y^2 = a^2$

$\vec{r} = (x(t))\hat{i} + (y(t))\hat{j}$.

$\left.\begin{array}{l} x = a\cos t \\ y = a\sin t \end{array}\right\}$ paramatrization for a circle.

$\left.\begin{array}{l} x = a\cos t \\ y = b\sin t \end{array}\right.$ ← ellips :-

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2 t + \sin^2 t$
$= 1$

$x = a\sec t$. Hyperbola.
$y = b\tan t$.

$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \sec^2 t \cancel{+} \tan^2 t = 1$

circle :-

$\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j}$

$\vec{v} = \frac{\vec{dr}}{dt} = (-a\sin t)\hat{i} + (a\cos t)\hat{j}$

$|\vec{v}| = |a| = a$.

$T = \frac{\vec{v}}{|\vec{v}|} = (-\sin t)\hat{i} + (\cos t)\hat{j}$

$\frac{dT}{dt} = -\cos t\,\hat{i} - \sin t\,\hat{j}$

$\frac{dT}{dt} = 1 \downarrow$

$K = \left(\frac{1}{a}\right)(1) = \frac{1}{a}$, حيث نقر

الدوائر كل ما زاد نصه بيكون سهل على الدائرة ،

لأنه K تقل .

Ex) $\vec{r}(t) = ((\cos t) + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j}$. $t > 0$.

$\vec{r}'(t) = (-\sin t + t\cos t + \cancel{\sin}\ t)\hat{i} + (t\sin t)\hat{j}$

$= (t\cos t)\hat{i} + (t\sin t)\hat{j}$

$|\vec{v}| = t$

$T = \dfrac{\vec{v}}{|\vec{v}|} = \cos t\, i + (\sin t)\hat{j}$

$\dfrac{d\vec{T}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j}$

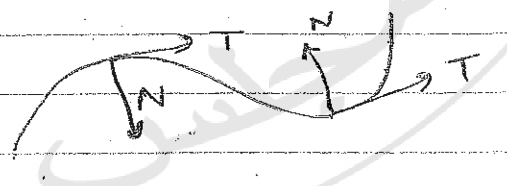
$\left|\dfrac{dT}{dt}\right| = 1$

$K = \dfrac{1}{|v|}\left(\dfrac{dT}{dt}\right)$

$K = \dfrac{1}{T}(1) = 1$

Def :- A point where $K \neq 0$, the priencipal unit normal vector for a smooth curve in plane

is $\vec{N} = \dfrac{1}{K}\dfrac{d\vec{T}}{ds}$



$N = \dfrac{\frac{dT}{ds}}{\left|\frac{d\vec{T}}{ds}\right|}$ ← unit vector $= \dfrac{\frac{dT}{dt}\,\cancel{\frac{dt}{ds}}}{\left|\frac{dT}{dt}\,.\,\cancel{\frac{dt}{ds}}\right|} = \dfrac{dT/dt}{dT/dt}$

$T = \dfrac{v}{|v|}$ $N = \dfrac{dT/dt}{|dT/dt|}$

$\cancel{Ns}$

$T \to \dfrac{dT}{dt} \to \left|\dfrac{dT}{dt}\right|$ بنجل .

$(r(t)) \Rightarrow \vec{T} = \frac{\vec{v}}{|\vec{v}|} \rightarrow K = \left|\frac{dT}{ds}\right| = \frac{1}{v^2}\left|\frac{d\vec{T}}{dt}\right|$

$\Rightarrow N = \frac{1}{K}\frac{dT}{ds} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$

Ex:- let $\vec{r}(t) = \ln(\sec t)\hat{i} + t\hat{j}$ $\quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

find $\vec{T}, K, \vec{N}$.

$\vec{v} = \left(\frac{1}{\sec t} \cdot \sec t \tan t\right)\hat{i} + \hat{j}$

$= \tan t\,\hat{i} + \hat{j}$

$|\vec{v}| = \sqrt{\tan^2 t + 1} = \sqrt{\sec^2 t} = |\sec t| = \sec t$

$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sec t}[\tan t\,\hat{i} + \hat{j}] = (\sin t)\hat{i} + (\cos t)\hat{j}$

$\frac{dT}{dt} = \cos t - \sin t\,\hat{j}$

$\left|\frac{dT}{dt}\right| = 1 \longrightarrow K = (\cos t)(1) = \cos t$

$N = \frac{\cos t\,\hat{i} - \sin t\,\hat{j}}{1} = (\cos t)\hat{i} - (\sin t)\hat{j}$

(2) $\vec{r}(t) = (e^t \cos t)\hat{i} + (e^t \sin t)\hat{j} + 2k$

$\vec{v} = (-e^t \sin t + e^t \cos t)\hat{i} + (e^t \cos t + e^t \sin t)\hat{j} + 0$

$|\vec{v}| = \sqrt{e^{2t}[\sin^2 t + \cos^2 t - 2\sin t\cos t + \cos^2 t + \sin^2 t + 2\sin t\cos t]}$

$= \sqrt{2e^{2t}} = \sqrt{2}\,e^t$

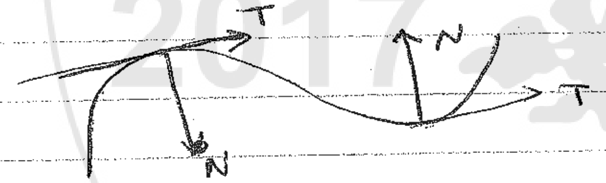
$$T = \frac{1}{\sqrt{2}\,e^t}\left[(-e^t\sin t + e^t\cos t)\,\hat{i} + (e^t\cos t + e^t\sin t)\,\hat{j}\right]$$

$$= \frac{1}{\sqrt{2}}\left[(-\sin t + \cos t)\,\hat{i} + (\cos t + \sin t)\,\hat{j}\right].$$

$$\Rightarrow \frac{dT}{dt} = \frac{1}{\sqrt{2}}\left[-(\cos t + \sin t)\,\hat{i} + (\cos t - \sin t)\,\hat{j}\right].$$

$$\left|\frac{dT}{dt}\right| = \frac{\sqrt{2}}{\sqrt{2}} = \boxed{1}$$

$\Rightarrow$ Circle of curvature of plane curve.



* The circle of curvature (oscilating circle) at a point $P(x,y)$. on plan ~~of the curve~~ where $K \neq 0$.

is acircle in plane of the curve that:

1) is atangent to curve ~~ap~~ at P.

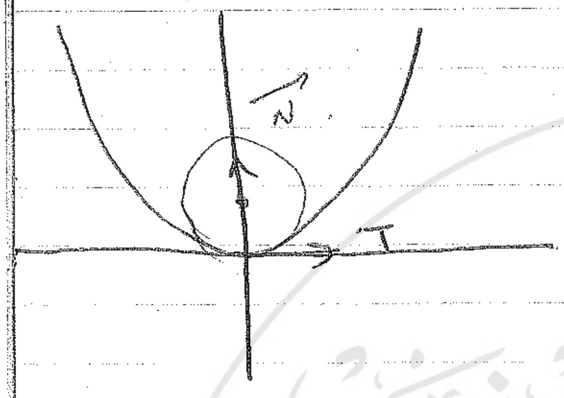
2) has the same curvature, the curve has at P

3) lies toward the concave of the curve.

* The raudius of the circle of curvature $\Rightarrow$

ملاحظة $\rho = \frac{1}{K}$

Ex) : Find and graph the circle of curvature for $y = x^2$ at the origen.



$x = t \qquad -\infty < t < \infty$

$y = t^2$

$\vec{r}(t) = t\hat{i} + t^2\hat{j}$

$\frac{dr}{dt} = \hat{i} + 2t\hat{j}$

$|\vec{v}| = \sqrt{1+4t^2}$

$T = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{1+4t^2}} \left[\hat{i} + 2t\hat{j}\right]$

$= (1+4t^2)^{-1/2}\hat{i} + (2t)(1+4t^2)^{-1/2}\hat{j}$

$\frac{dT}{dt} \Rightarrow k(0) = 2$

$\rho = \frac{1}{k} = \frac{1}{2} \rightarrow$ circle center $(0, \frac{1}{2})$

Equat :- $x^2 - (y - \frac{1}{2})^2 = 1/4$

$\vec{N} = \hat{j}$

$\vec{T} = \hat{i}$

13.5 :- Tangential and Normal Componanat of Acceleration.

$a : T + N$

$$\vec{a} = (\quad)\vec{T} + (\quad)\vec{N}$$

$\vec{T}, \vec{N}, \vec{B}$ FRAM.

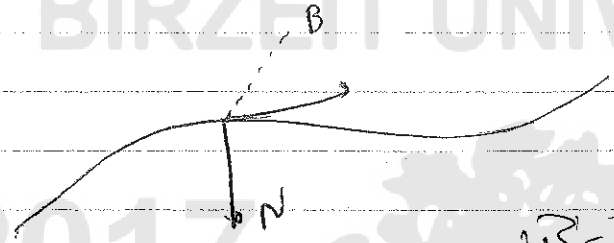
Tangent, Normal, Binormal

$a = T N$

Def :- The Binormal vector of curve $(\vec{r(t)})$ in Space is The vector $\vec{B} = \vec{T} \times \vec{N}$

$B = T \times N$

N, T عمودي على B.



$B = T \times N$

Note :-

1) $|B| = 1$
2) $\vec{B}$ is orthogonal (عمودي) to both $\vec{T}$ and $\vec{N}$.
3) $\vec{T}, \vec{N}$ and $\vec{B}$ define a right-hand vector fram.

TNB fram or Frenet Fram.

Ex : $\vec{r}(t) = (t+1)\hat{i} + (2t)\hat{j} + t^2\hat{k}$, Find $T, \vec{N}, \vec{B}$ Fram.

$\vec{r}' = \hat{i} + 2\hat{j} + 2t\,k$.

$|\vec{r}'| = \sqrt{5+4t^2}$

$$T = \frac{1}{\sqrt{5+4t^2}}\left[\hat{i} + 2\hat{j} + 2t\,k\right].$$

$= (5+4t^2)^{-1/2}\hat{i} + (2(5+4t^2)^{-1/2}\hat{j} + 2t\ (5+4t^2)^{-1/2}\hat{k}$

$\frac{dT}{dt} = (-\frac{1}{2})\ (5+4t^2)^{-3/2} \cdot (8t)\ \hat{i} + \cdots$

$N =$

Ex) $\vec{r}(t) = (\cosh(t))\hat{i} - (\sinh t)\hat{j} + t\hat{k}$.
Find T, N, B, FRAM.

$N = \frac{1}{k} |\partial T/\partial t|$

$N = \frac{\partial T/\partial t}{|\partial T/\partial t|}$

Tangent and Normal componats of Acceleration.

$$\vec{a} = a_{\vec{T}}\, \vec{T} + a_{\vec{N}}\, \vec{N}$$

$$\Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} = T \cdot \frac{ds}{dt}$$

$$\frac{dr}{ds} = \frac{d\vec{r}}{dt} \frac{dt}{ds} \quad \frac{1}{\frac{ds}{dt}} \quad = \frac{\vec{v}}{|\vec{v}|} = \vec{T}$$

$$\vec{a} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\vec{T} \frac{ds}{dt}\right)$$

$$a_T = |v(t)|'$$

$$a_N = \sqrt{a^2 - (a_T)^2}$$

$$= \vec{T}\left(\frac{d^2 s}{dt^2} + \frac{dT}{dt} \cdot \frac{ds}{dt}\right.$$

$$\left(\frac{d^2 s}{dt^2}\right)\vec{T} + k\left(\frac{ds}{dt}\right)\vec{N}$$

$$aT = \left(\frac{d^2 s}{dt^2}\right) = \frac{d}{dt}|\vec{v}| \quad , a_N = k|\vec{v}|^2$$

$$= \frac{1}{\rho}|\vec{v}|.$$

$$\frac{dT}{ds}\left(\frac{ds}{dt}\right)^2 \quad k\vec{N}$$

$$\frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt} . \quad N = \frac{1}{k}\frac{dT}{ds} \Rightarrow \frac{dT}{ds} = k N.$$

$\vec{a} = a_{\vec{T}} \vec{T} + a_{\vec{N}} \vec{N}$. $\sqrt{\ }$)a

$\vec{a} \cdot \vec{a} = |\vec{a}|^2 = (a_T)^2 + (a_N)^2$.

$$\boxed{\vec{a}_{\vec{N}} = \sqrt{|a|^2 - (\vec{a}_T)^2}}$$

Ex):- $r = (\cos t + t \sin t)\hat{i} + \sin t - t\cos t)\hat{j}$. $t \geq 0$.

Find $\vec{a}_N$, $\vec{a}_T$

$\vec{v} = (-\sin t + \sin t + t\cos t)\hat{i} + (\cos t - \cos t + t\sin t)\hat{j}$.
$= (t\cos t)\hat{i} + (t\sin t)\hat{j}$.

$|\vec{V}| = t$.

$\frac{d}{dt}(\vec{v}) = \cancel{t}\ 1$

$\vec{a} = (\cos t - t\sin t)\hat{i} + (t\cos t + \sin t)\hat{j}$.

$|\vec{a}| = \sqrt{\cos^2 t - \cancel{2\sin t \cos t} + t^2\sin^2 t + \sin^2 t + \cancel{2t\sin t\cos t} + t^2 + \cos^2 t}$

$= \sqrt{1 + t^2}$

$a_{\vec{N}} = \sqrt{(1+t^2) - (1)^2} = \sqrt{t^2} = t$

$\vec{a} = 1 \cdot T + t\vec{N}$.

Torsion.

$B = T \times N$

$\vec{B} = (\vec{T}) \times (\vec{N})$

$\frac{d\vec{B}}{ds} = \frac{d}{ds}(\vec{T} \times \vec{N})$

$= \frac{dT}{ds} \times \vec{N} + T \times \frac{dN}{ds}.$

$0 + \vec{T} \frac{dN}{ds}$

$\frac{dB}{ds} = \vec{T} \times \frac{d\vec{N}}{ds}$

* $\frac{dB}{ds}$ is orthogonal to $\vec{T}$.

* $\frac{dB}{ds} \Rightarrow$ is " to $\vec{B}$

$\frac{dB}{ds}$ is parallel to $\vec{N}$.

$\frac{dB}{dB} = K \vec{N}$

$\frac{d\vec{\beta}}{ds} = -Z \vec{N}$ (tau).

$\tau = -N \cdot \frac{dB}{ds}$

$\frac{dB}{ds} \cdot \vec{N} = -Z \vec{N} \cdot \vec{N}.$

$\Rightarrow \tau$ is called the torsion.

1) $T = \dfrac{\vec{v}}{|\vec{v}|}$. tangent unit vector.

2) $\vec{N} = \dfrac{d\vec{T}/dt}{|dT/dt|}$. principal Normal vector.

3) $\vec{B} = \vec{T} \times \vec{N}$.

4) $k = \left|\dfrac{dT}{ds}\right| = \dfrac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$ $\quad \dfrac{1}{|v|} |dT/dt|$

$\left|\dfrac{dT}{ds}\right| , \left|\dfrac{\vec{v} \times \vec{a}}{|v|^3}\right|$

5) $\tau = -\dfrac{dB}{ds} \cdot \vec{N}$

$$= \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2}$$

$\dfrac{1}{\overline{v \times a}}$

6) $\vec{a} = a_{\vec{T}} \vec{T} + a_{\vec{N}} \vec{N}$

$a_{\vec{T}} = \dfrac{d}{dt} |\vec{v}|$

$a_{\vec{N}} = k |\vec{v}^2| = \sqrt{|a|^2 - |a_{\vec{T}}|^2}$ $\quad v^5 \quad \dfrac{|v \times a|}{|v|^3}$

9/738 $r(t) = (3\sin t)\hat{i} + (3\cos t)\hat{j} + 4t\,k.$

$\vec{v} = (3\cos t)\hat{i} - (3\sin t)\hat{j} + 4$

$\vec{a} = (-3\sin t)\hat{i} - (3\cos t)\hat{j}$

$$\vec{r} \times \vec{a} = \begin{vmatrix} i & j & k \\ 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \end{vmatrix}$$

$(12\cos t)i + (12\sin t)j + 9(\cos t - 9\sin t^2)\,k$

$$\tau = z = \frac{\begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \\ -3\cos t & +3\sin t & 0 \end{vmatrix}}{(15)^2} = \frac{-36}{225}$$

$(-9\cos^2 t - 9\sin^2 t)\,k + (12\cos t)\,i + (12\sin t)\,j$

$-9\,k + \ldots$

$-9\,k + (12\cos t)\,i + (12\sin)\,j$

$\sqrt{(-9)^2 + (12\cos t)^2 + 12\sin^2}$

$\sqrt{(9)^2 + (12)^2 + (12)^2}$

$= \sqrt{81 + 144}$

$= 15$

$(15)^1$

Ch 14 | 14.1 Function of several variable :-

$y = f(x)$
Dep — indep.

* we will foucuse on two variable.

$z = f(x, y) \Rightarrow R^2$
Dep — indep.

$w = f(x, y, z) \Rightarrow R^3$
$\vdots$
R^n . n-tuples.

Def :- Suppose that D is the set of n-tuples $(x_1, x_2, \ldots, x_n)$ $(D = R^n)$, $x_1, x_2 \ldots x_n \in R$.
A real valued function f on D is a rule that assigns a unique real number $f(x_1, x_2, \ldots, x_n)$ for each $(x_1, x_2, \ldots, x_n) \in D$.

$$f : D \longrightarrow \mathbb{R}$$

$$f(x_1, x_2, x_3 \ldots x_n) = w \in \mathbb{R}$$

w = dependent varriable.
$x_1, x_2, x_3 \ldots x_n$ independent variable.
D = Domain of the function
R = the range " "

Ex: $f(x,y) = \sqrt{x-y^2}$

$f(5,1) = \sqrt{5-1} = \sqrt{4} = 2$

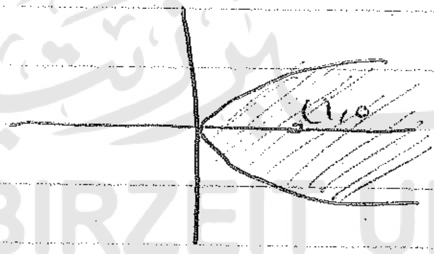
$(5,1) \longrightarrow 2$

* What is the Domain ?

$x-y^2 \geq 0 \longrightarrow \underline{x \geq y^2}$ $\qquad x \geq y^2$

$x - y^2 \geq 0$

$x \geq y^2$



Test point : $(1,0)$

$1 \geq 0$ ✓

Ex $\boxed{1}$: $z = f(x,y) = \ln\left(\frac{x}{y}\right)$

D: $\{(x,y) : xy > 0,\ y \neq 0\}$

Rang : $(-\infty, \infty)$

$(-\infty, 0) \cup (0, \infty)$ $\qquad \ln 0$ ∞

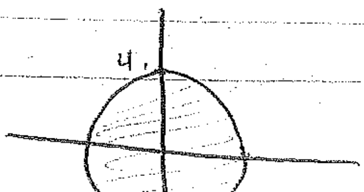
Ex $\boxed{2}$: $f(x,y) = \sqrt{16 - x^2 - y^2}$ $\qquad x^2 + y^2 < 16$

$z \geq 4 \qquad = \sqrt{16 - (x^2 + y^2)}$ $\qquad 16 - x^2 - y^2 \geq 0$

$x^2 + y^2 < 16$

D: $16 - (x^2 + y^2) \geq 0$. disk with center $(0,0)$ and $r = 4$

Range $= [0, 4]$.



Ex $\boxed{3}$:- $f(x,y) = \ln(x^2+y^2)$.

$D = \{ (x,y) : (x,y) \neq (0,0) \}$

or

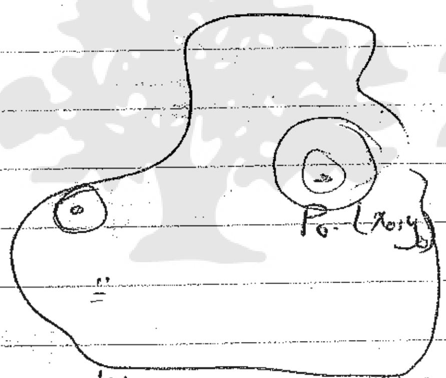
$R^2 \setminus \{(0,0)\}$.

Range $= (-\infty, \infty)$.

Definition:-

① A point (x_0, y_0) in a region R in the xy-Plane (2D) is called an <u>interior</u> point of R if it is a center of a disk of positive raduse that lies entirely in R.

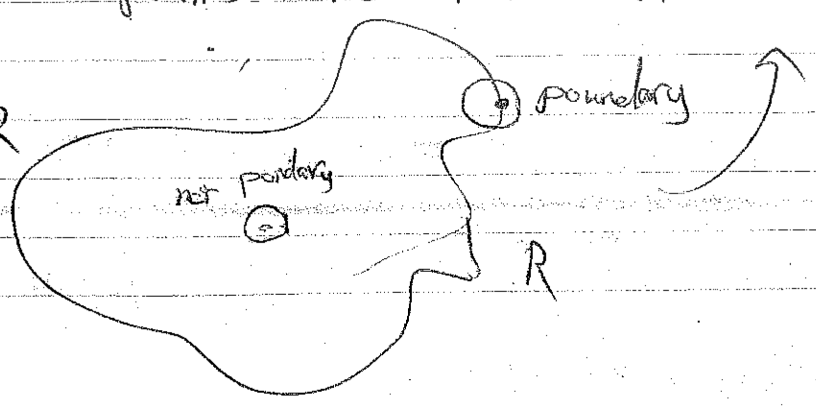
نقطة داخلية



نقطة طرفية اذا اخذت disk رح يطلع نقاط داخل R ونقاط برا R.

② A point (x_0, y_0) is <u>a boundary</u> point of R, if every disk centered at (x_0, y_0) contains points that lie outside R as points lie inside R.

boundary points $\notin R$ or $\in R$



③ The interior of regon R. is the set of all interior points of R.

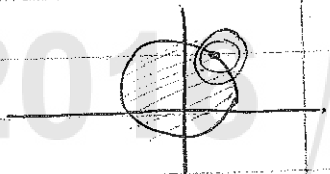
$Int(R) = \{ (x,y) : (x,y)$ is an interior point.

④ The region boundary is the set of all boundary points of this regon.

⑤ A region R is called open regoin if it consists only of interior points.

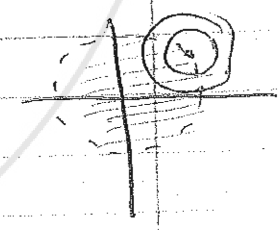
⑥ A region R is called ~~of~~ closed if it contains all it's boundary points.

Ex1:- ① $x^2+y^2 \leq 16$



boundary : $x^2+y^2=16$.
closed.

② $x^2+y^2 < 16$ boundary

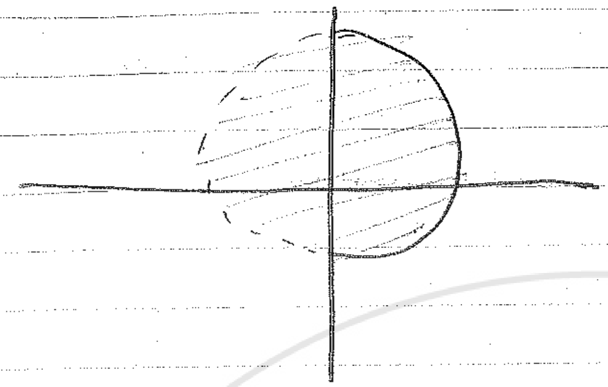


boundary = $x^2+y^2=16$.

boundary $\notin$ R $\Rightarrow$ Not closed.

interior: $x^2+y^2 < 16$.
$\Rightarrow$ open.

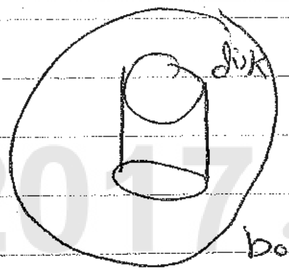
③



neither open
nor close.

Def :- A region R in the plane is bounded if it lies insid a disk of fixed raduse. نق موجود

Ex.



→ triangel.

not ~~b~~ bounded ⇒ line ⟷

مثل R^2.

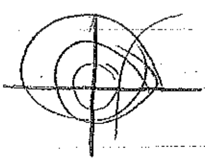
x-axis.

Ex: $f(x,y) = \ln(x^2+y^2)$

$D: \{(x,y) : (x,y) \neq (0,0)\}$.

open , Not bounded.

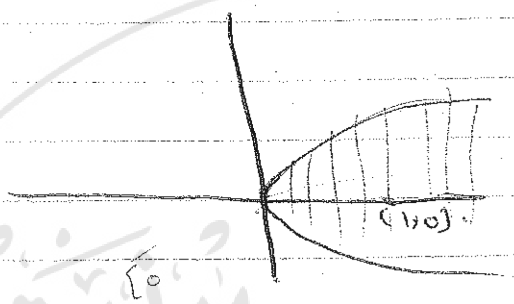
(0,0) is a boundary point



② $f(x,y) = \sqrt{x - y^2}$

$D = \{ (x,y) : x - y^2 \geq 0$

$\Rightarrow y^2 \leq x$

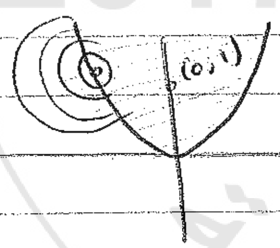


not bounded. $x = y^2 \in D \Rightarrow$ Closed.

Ex] $f(x,y) = \dfrac{1}{\sqrt{y - x^2}}$

$D : \{ (x,y) : y - x^2 > 0 \}$

$x^2 > y^2$



D:- Not bounded.
boundaries:- $y = x^2 \notin D$.
Not closed.
(open).

Def:- ① $\{ (x,y) : (x,y) \in D$ and $f(x,y) = c$
where c is constant $\}$
is called the level curve of $f(x,y)$.

$z = f(x,y) \Rightarrow$ 3D.

$f(x,y) = c$ (constant) $\Longleftrightarrow$ level curve 2D

$\rightarrow \in$ Range.

Def: ② : $\{ x, y, f(x,y)) : (x,y) \in D f \}$ is
called the graph of $f(x,y)$. The

graph of $z = f(x,y)$ is called the surface of $f(x,y)$.

Ex:- $f(x,y) = 100 - x^2 - y^2$.
find the level curve at
$c = 51, 75, 25$

$D \equiv$ entire plane.
Range $\equiv$
$z = 100 - (x^2 + y^2)$. or $x^2 + y^2 = 100 - z$.

Range : $z \leq 100$.
$\boxed{c = 51}$ $f(x,y) = 51$

$51 = 100 - (x^2 + y^2)$
$x^2 + y^2 = 49$.
circle

$c = 75 \Rightarrow x^2 + y^2 = 25$. circle

$c = 25 \rightarrow x^2 + y^2 = 75$. circle.
$c = 0 \rightarrow x^2 + y^2 = 100$.
$c = 100 \rightarrow x^2 + y^2 = 0$ original.
level curve $\equiv$ circles

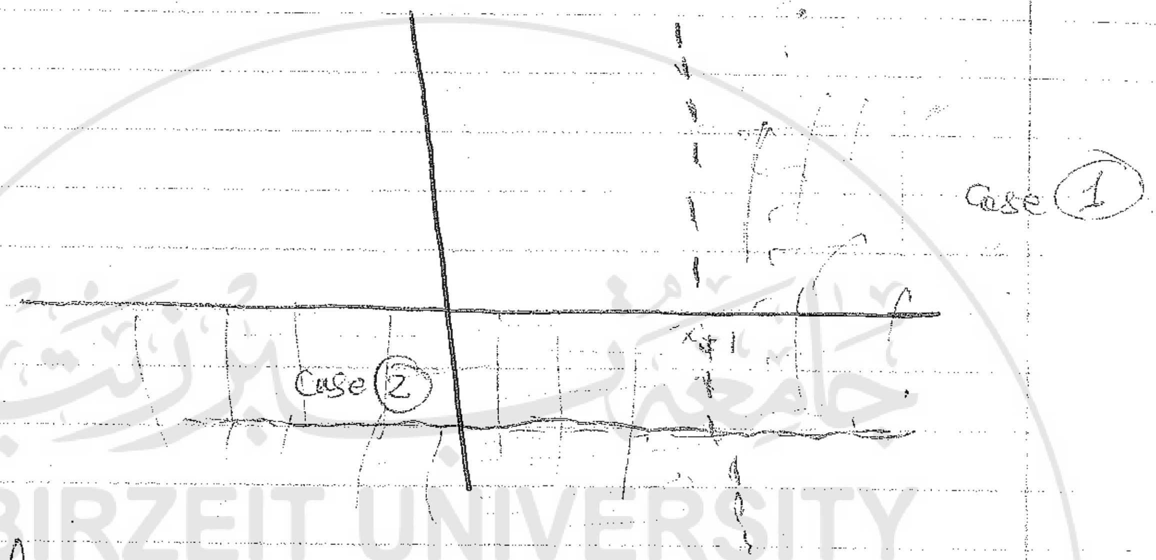
$\boxed{10}$ $f(x,y) = \ln(xy + x - y - 1)$
$xy + x - y - 1 > 0$.
$x(y+1) - (y+1) > 0$

$(y+1)(x-1) > 0$.

$y = -1$ $x_{>c}$

Case 1 $x-1>0$ and $y+1>0$

Case 2 $x-1<0$ and $y+1<0$



boundes :- $x=1$ $y=-1$

Not closed Not bounded

$$f(x,y)=c \longrightarrow \ln\big((x-1)(y+1)\big)=c$$

$$e^c=(x-1)(y+1)$$

$$y+1=\frac{e^c}{x-1}$$

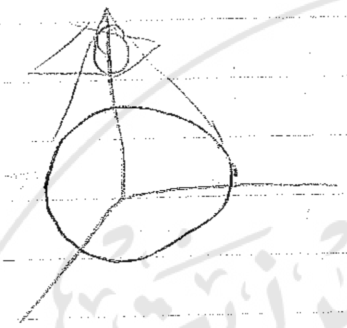
$$\boxed{y=\frac{e^c}{x-1}-1}$$

Def 3 :- The curve in space in wich the plane $z=c$ cut the surface $z=f(x,y)$ is called the contour curve $f(x,y)=c$.

$z = f(x,y) = 100 - (x^2 + y^2)$

$z = 75$

$\Rightarrow f(x,y) = 75 \longrightarrow x^2 + y^2 = 25$ circle



Def (4) :- Function of 3 variables :-

$w = f(x,y,z) \longrightarrow 4D.$

Def :- $\{(x,y,z) : f(x,y,z) = c, c \in \text{constant}\}$ is called a level surface.

Ex :- $w = f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$

* level surface :-

$x^2 + y^2 + z^2 = c^2$. Sphere.

Ex :- $f(x,y,z) = z - x^2 - y^2$.

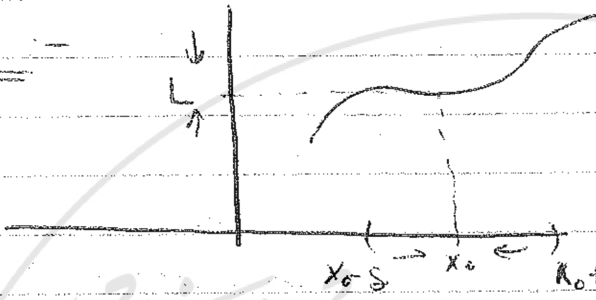
level surface :-

$z - (x^2 + y^2) = c.$

$x^2 + y^2 = z - c$

$\boxed{14.2}$ Limits and continuity in Higher Dimension:-

2D :-



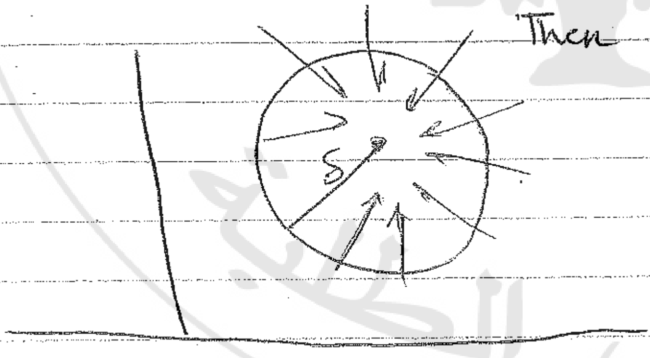
$$\lim_{x \to x_0} f(x) = L$$

3D :- $\longleftrightarrow$ Intival :- $\longrightarrow$ Disk.

Let $z = f(x,y)$ $\quad P_0(x_0, y_0)$.

$f(x,y)$ approches L as $P(x,y)$ approches $P_0(x_0, y_0)$.

$$\lim_{P(x,y) \to (x_0,y_0)} f(x,y) = \lim_{P \to P_0} f(x,y) = L$$



Then the limit exists.

infinitle ~~#~~ of ~~P~~ pathes.

Ex:- ① $\lim_{\substack{P \to P_0 \\ (x,y) \to (x_0,y_0)}} x = x_0$

② $\lim_{x,y \to (x_0,y_0)} y = y_0$

③ $\lim k = k$

Theorem:-

If $\lim f(x,y) = L$.
$\lim g(x,y) = M$ then:-

1) $\lim f(x,y) \pm g(x,y) = L \pm M$.

2) $\lim K f(x,y) = K L$.

3) $\lim f(x,y) \cdot g(x,y) = L \cdot M$

4) $\lim \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M} \quad M \neq 0$.

5) $\lim\limits_{P \to P_0} (f(x,y))^n = L^n$.

Ex: 1) $\lim\limits_{\to(0,0)} \cos\left(\dfrac{x^2+y^2}{x+y+1}\right) = \cos\left(\dfrac{0}{1}\right) = 1$

2) $\lim\limits_{P \to (0} \dfrac{\cos y + 1}{y - \sin x} = \dfrac{0+1}{\pi/2} = \dfrac{1}{\pi/2} = \dfrac{2}{\pi}$

3) $\lim\limits_{P \to (0,0)} \dfrac{xy - x^2}{\sqrt{y} - \sqrt{x}} = \left(\dfrac{0}{0}\right)$,

بالمرافق
ما بنضرب ~~بالعكس~~ با إذا
نحل كالتالي.

$\lim\limits_{P \to (0,0)} \dfrac{x\,(y-x)}{\sqrt{y} - \sqrt{x}} = \lim \dfrac{x(\sqrt{y}-\sqrt{x})(\sqrt{y}+\sqrt{x})}{(\sqrt{y}-\sqrt{x})} = 0$

4) $\lim\limits_{P \to (2,-4)} \dfrac{y+4}{x^2y - xy + 4x^2 - 4x}$. $\left(\dfrac{0}{0}\right)$

$y \neq -4 \Rightarrow x \neq x^2$

$\lim \dfrac{(y+4)}{x^2(y+4) - x(y+4)} = \lim \dfrac{\cancel{y+4}}{\cancel{(y+4)}(x^2 - x)} = \dfrac{1}{2}$.

Continuity :-

Def :- The function $f(x,y)$ is continous at $P_0:(x_0,y_0)$ if :-

1) $f(x_0,y_0)$ is defined.
2) $\lim_{P\to P_0} f(x,y)$ exists.
3) $\lim_{P\to P_0} f(x,y) = f(x_0,y_0)$.

Note :- ① $f(x,y)$ is a continuous functions if it is cont. at each point of it's Domain.

② Polynomials, Rational functions are cont.

* Ex : $z = f(x,y) = \ln(x^2+y^2)$
  Cont for all $(x,y) \neq (0,0)$

* Ex :- $z = f(x,y) = \dfrac{x+y}{2+\cos xy}$.

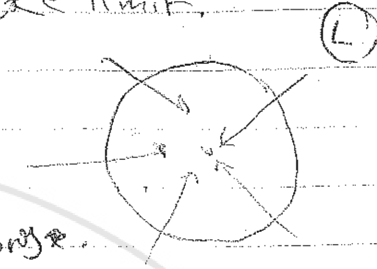
  Cont at all (x,y) in R^2.

* Ex :- a) $f(x,y,z) = \ln(xyz)$.
  Cont for All (x,y,z) ; $xyz > 0$.

* Ex :- $f(x,y,z) = e^{x+y}\cos z$.
  Cont for All $(x,y,z) \in R^3$.

Two Path test for None existenke limit.

كل Path لازم يعطي نفس الرقم



Test: if a function $f(x,y)$ have different limits alonge two different paths as $(x,y) \Rightarrow (x_0,y_0)$

$\Rightarrow \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$. D.N.E .

Ex] :– $f(x,y) = \dfrac{2x^2y}{x^4+y^3}$ find $\lim\limits_{p\to(0,0)} f(x,y) = \left(\frac{0}{0}\right)$

نختار مسار ① $y = x \Rightarrow \lim\limits_{\substack{p\to(0,0)\\ y=x}} \dfrac{2x^3}{x^4+x^3} = \lim\limits_{p\to(0,0)} \dfrac{2x^3}{x^3(x+1)}$

$= \boxed{2}$

path $y=-x$. $\lim \dfrac{-2x^3}{x^4-x^3} = 2$ [illegible]

Path y – axis $(0,y)$. نعوض

$\lim \dfrac{0}{y^3} = 0$. So $\lim\limits_{p\to(0,0)} f(x,y)$. D.NE .

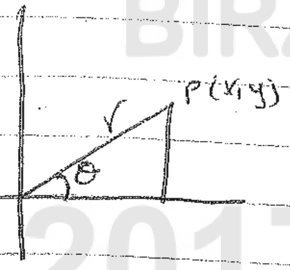
Ex] $\frac{44}{762}$ $f(x,y) = \frac{xy}{|xy|}$, $\lim_{p \to (0,0)} f(x,y)$.

$= \frac{xy}{|y||x|}$

path 1 $\boxed{x = y}$ $\lim \frac{x^2}{x^2} = \boxed{1}$.

path 2 $\boxed{x = -y}$ $\lim \frac{x^2}{-x^2} = \boxed{-1}$ L.D.E.

When $p = (0,0)$ → نحل بهاي الطريقة ←



$x = r\cos\theta$.
$y = r\sin\theta$.

(polar Coordinates)

$f(x,y) \Rightarrow \lim_{r \to 0} f(r\cos\theta, r\sin\theta)$.

$P_0(0,0)$.

Ex] $f(x,y) = \frac{x^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2} = \lim_{r \to 0} r\cos^3\theta$
$= 0$.

Ex] $f(x,y) = \frac{2x}{x^2+x+y^2}$

$\lim f(x,y) = \lim \frac{2r\cos\theta}{r^2\cos\theta^2 + \sin^2\theta + r\cos\theta}$

$= \lim \frac{2r\cos\theta}{r^2 + \cos\theta} = 2$

$$g(x,y) = \frac{2x}{x^2+y^2+y}$$

$$\lim \frac{2\cos\theta}{r+\sin\theta} = 2\cot\theta$$

$\Rightarrow$ D.N.E.

$\boxed{14.3}$ partial Derivatives.

$z = f(x,y)$, $D \equiv$ Domain of $f(x,y)$.

$y = y_0$ plane ($\parallel$ xz plane, $\perp$ y-axise).

$y = y_0$ and $z = f(x,y)$

$z = f(x, y_0)$. Curve in plane

$$y = f(x).$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x_0+h) - f(x)}{h}$$

Df :- The first partial derivative of $z = f(x,y)$ at $P_0(x_0, y_0)$, with respect to x.

is, $$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

provided the limit exists.

Note :- ① $\left.\frac{df}{dx}\right|_{(x_0,y_0)} = \frac{d}{dx}(f(x, y_0)$

② Notations :-

$$\frac{df}{dx}\Big|_{P_0} = \frac{df}{dx}(x_0, y_0) = \frac{dz}{dx}\Big|_{P_0} = z_x = f_x$$

③ $\frac{df}{dx}\Big|_{P_0}$ = The slope of the curve $z = f(x,y)$ at the point $(x_0, y_0, f(x_0, y_0))$

④ $\frac{df}{dx}$ = The rate of change of $f(x,y)$ with respect to x while $\underline{y}$ is constant.

$\boxed{\text{Def}}$: $$\frac{df}{dy}\Big|_{P_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

$$\frac{df}{dy} = \frac{dz}{dy} = z_y = f_y$$

$\frac{60}{773}$

$$f(x,y) = \begin{cases} \frac{\sin(x^3 + y^4)}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$f_x = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$\lim \frac{\sin h^3}{h^2} = \lim_{h \to 0} \frac{\sin^3}{h^3} = 1$$

$$f_y = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h^x}{h^x} \cdot \frac{\sin h^n}{h^3} = 0.$$

Ex) find f_x & f_y

1) $f(x,y) = y \sin xy$

$f_x = y(\cos xy)(y) = y^2 \cos xy.$

$f_y = y(x \cos xy) + \sin xy.$

2) $f(x,y) = x^3 + 2x^3y^4 + y^4 + 10$

$f_x = 3x^2 + 6y^4x^2 + 0 + 0.$

$f_y = 0 + 2x^3 \cdot 4y^3 + 4y^3 + 0$

$= 8x^3y^3 + 4y^3.$

3) $f(x,y) = (x^2 + y^3)^{10}.$

$f_x = 6(x^2 + y^3)^9 \cdot 2x.$

$f_y = 10(x^2 + y^3)^9 \cdot 3y^2$

Ex) Find z_x, z_y for:-

$$yz - \ln z = x + y$$

$$z(x,y) - \ln(z(x,y)) = x + y$$

$$y z_x - \frac{1}{z} z_x = 1 + 0$$

$$z_x \left(y - \frac{1}{z}\right) = 1 \rightarrow z_x = \frac{1}{y - 1/z}$$

* $z_y = y \cdot z_y + z \cdot 1 - \frac{1}{z} z_y = 1$

$$z_y \left(y - \frac{1}{z}\right) = 1 - z$$

$$z_y = \frac{1 - z}{y - \frac{1}{z}}$$

$\boxed{14.3}$ Partial Derivative:-

* 1^{st} PD
  f_x, f_y,

* 2^{nd} order PD
  If $z = f(x,y)$, then

$$\frac{d^2 z}{dx^2} = \frac{d^2 f}{dx} = f_{xx} = z_{xx} = \frac{d}{dx}(f_x)$$

The second PD withe respect to x

$$\frac{d^2z}{dy^2} = f_{yy} = \frac{d}{dy}(f_y)$$

Mixed PD :-

* $f_{xy} = \frac{d}{dy}(f_x)$

$f_{yx} = \frac{d}{dx}(f_y)$

Ex] $w = e^x + x\ln y + y\ln x$

$w_x = e^x + \ln y + \frac{y}{x}$

$w_{xx} = e^x - \frac{y}{x^2}$

$w_{xy} = \frac{d}{dy}(w_x)$

$= \frac{1}{y} + \frac{1}{x}$

$w_y = \frac{x}{y} + \ln x$

$w_{yy} = \frac{-x}{y^2}$

$w_{yx} = \frac{d}{dx}(w_y) = \frac{1}{y} + \frac{1}{x}$

The Mixed Derivative Theorem:-

If $f(x,y)$ and it's partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined through open regon continouse (a,b) and all are cont. at (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$.

Theorem:-

suppose that the 1st p.d. of $f(x,y)$ are defined on R. containing the point (x_0,y_0). and that f_x, f_y are cont. at (x_0,y_0).

Then $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$ satisfies the following equation

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

as $\Delta x, \Delta y \to 0$ then $\varepsilon_1, \varepsilon_2 \to 0$

Def:-

$f(x,y)$ is diff. if Δz satisfied the equation $(*)$.

Corally:-

If f_x, f_y are cont. through on open ~~Int~~ regon R then $f(x,y)$ is diff.

Theorem:-

If $f(x,y)$ is diff. at (x_0,y_0), then $f(x,y)$ is cont. at (x_0,y_0)

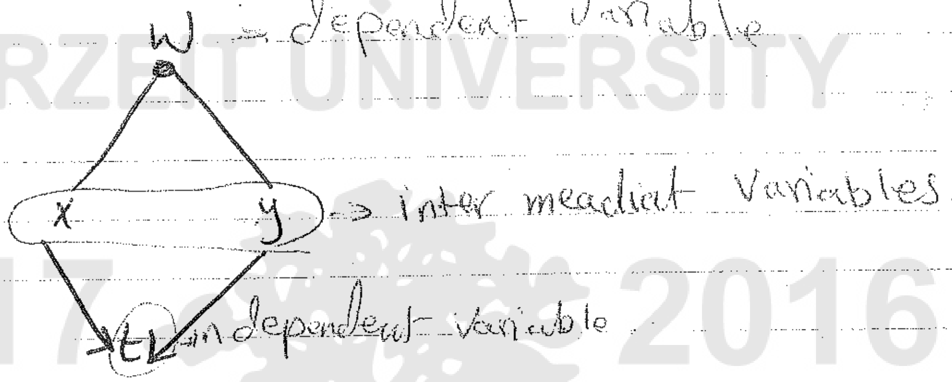
14.4 The Chain Rule

Theorem:- If $w = f(x,y)$, is diff function of ~~x,y~~ t, and $x = x(t)$ and $y = y(t)$. Then:

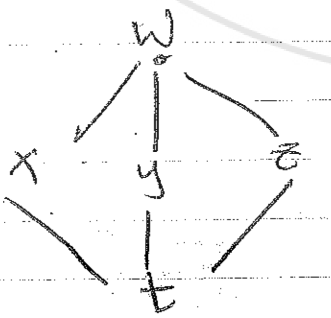
$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}.$$

---

* Tree Diagram.



$w = f(x, y, z)$, $x = x(t)$
$y = y(t)$.
$z = z(t)$.



$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt} + \frac{dw}{dy} \cdot \frac{dy}{dt} + \frac{dw}{dz} \cdot \frac{dz}{dt}$$

Ex) :- If $f(x,y) = x^2 + y^2$.

$x = t + \cos t$.

$y = t^2$.

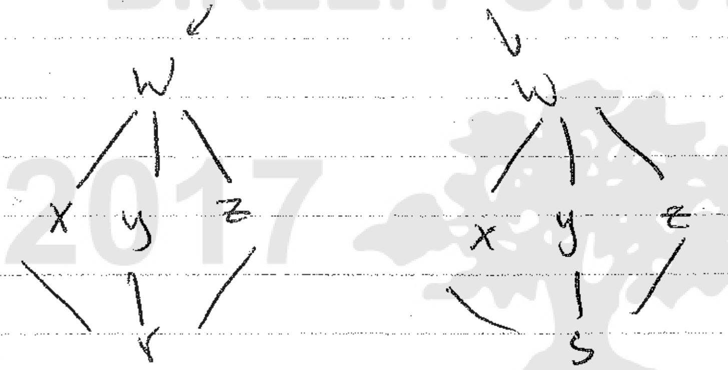
$$\frac{dw}{dt} = (2x)(1-\sin t) + (2y)(2t).$$

$$= 2(t+\cos t)(1-\sin t) + (2t^2)(2t)$$

Ex) Let $w = f(x,y,z)$.

$x = x(r,s)$, $y = y(r,s)$ $z = z(r,s)$.

$$\frac{\partial w}{\partial r} \; , \; \frac{\partial w}{\partial s}$$

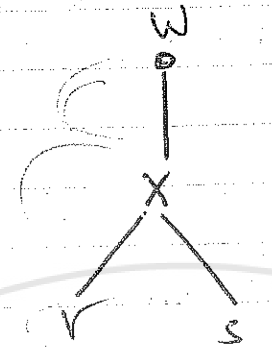


$$\ast \; \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$\ast \; \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \dots$$

$w = f(x)$, $x = g(r,s)$.

Ex



بارشل لانها بعدها مطولة

$$\frac{\partial w}{\partial r} = \frac{dw}{dx} \cdot \frac{\partial x}{\partial r} \Big\} \text{متغيرين}$$

مش بارشل لانه مشتقة لمتغير واحد.

$$\frac{\partial w}{\partial s} = \frac{dw}{dx} \cdot \frac{\partial x}{\partial s}$$

Ex

$w = ye^x \ln y$ $\quad (w = f(x, y)$

$x = \ln(r\cos\theta)$ $\quad\quad y = r\sin\theta$

$= x(r, \theta)$ $\quad\quad = y(r, \theta)$

$w \equiv$ dep. $\quad x, y \equiv$ int. $\quad r, \theta \equiv$ indep.

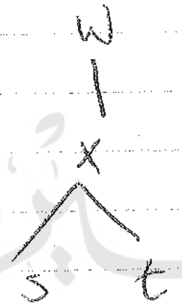
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= (y\, e^x \ln y) \frac{1}{r\cos\theta} \cdot \cos\theta + \frac{y\, e^x}{y} \cdot \sin\theta$$

$w = f(s^3+t^2)$, $f(x) = e^x$, Find $\frac{\partial w}{\partial t}$, $\frac{\partial w}{\partial s}$.

$w = f(x)$, $x = s^3 + t^2$.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} = e^x \cdot 3s^2$$
$$= e^{s^3+t^2}\cdot 3s^2.$$



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} = e^x \cdot 2t = e^{s^3+t^2}\cdot 2t.$$

Implicit Differentiation

$y = f(x)$

$x^2 y + y^2 + x^2 = 10y + 10x^4 y^5$

$F(x,y) = 0$

$W = F(x,y) = F(x, y(x))$

$W = F(x,y)$

$$\frac{dw}{dx} = F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = 0$$

$$F_x + F_y \frac{dy}{dx} = 0$$

$\frac{d}{dx} / F(x,y))$

$$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Ex ① $F(x,y) = x^2y + y^3 + x^2 - 10y - 10x^4y^5$

$F_x = 2xy + 2x - 40x^3y^5$

$F_y = x^2 + 3y^2 - 10 - 50x^4y^4$

$$\frac{dy}{dx} = -\frac{2xy + 2x - 40x^3y^5}{x^2 + 3y^2 - 10 - 50x^4y^4}$$

② $xe^y + \sin xy - y = \ln 2$

$F(x,y) = xe^y + \sin xy - y - \ln 2$

$F_x = e^y + y\cos xy$

$F_y = xe^y + x\cos xy - 1$

$$\frac{dy}{dx} = -\frac{e^y + y\cos xy}{xe^y + x\cos xy - 1} = \frac{-2 + \ln 2}{-1} = 2 + \ln 2$$

$$\frac{dy}{dx} = \frac{xe^y + x\cos xy - 1}{\cancel{x}e^y + y\cos xy}$$

14.4, 14.5 . 14.6.

quize :-

$f(x,y) = \sin^{-1}(x^2 - y^2)$.

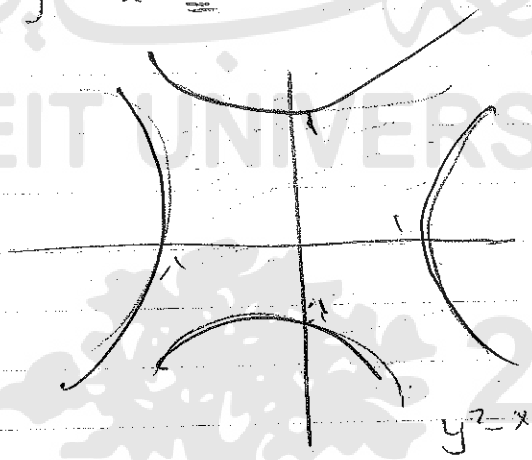
Range : $[-\pi/2 , \pi/2]$

$-1 \le x^2 - y^2 \le 1$

$x^2 - y^2 \le 1$ and $x^2 - y^2 \ge -1$
$y^2 - x^2 \le 1$

$x^2 - y^2 = 1$, $y^2 - x^2 = 1$

Hyp. -



$x^2 - y^2 = 1$

$y^2 - x^2 = 1$

bounded . : $x^2 - y^2 = \pm 1$

level curve . $\sin^{-1}(x^2 - y^2) = c$

level curve c $x^2 - y^2 = \sin c$.

hyperbola.

14.5 : ~~Direiative~~ Derivatives and The Gradient Vector.

Directional Derivative: The rate of change of function of 2 or more variable in any direction.

$z = f(x(t), y(t)) \Rightarrow \left.\frac{df}{dt}\right|_{P_0} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} =$

rate of change of $f(x,y)$ with respect to t.

* If $\vec{u} = u_1\hat{i} + u_2\hat{j}$ is any unit vector we want to find the rate of $f(x),y)$ in the ~~rate~~ direction $f\ \vec{u}$

let $f(x,y)$ be defined on a regon R (xy plane).

let $P_0\ (x_0, y_0) \in R$, $\vec{u} = u_1 i + y_2 j$ (unit vec

$\Rightarrow \begin{array}{l} x = x_0 + s u_1 \\ y = y_0 + s u_2 \end{array}$ } paramatrization of the line through $P_0(x_0, y_0)$ and parallel to $\vec{u}$

$S \equiv$ arc length..

Def:- The derivative of $f(x,y)$ at $P_0(x_0,y_0)$. in the direction of $\vec{u}$ is the number

$\left(\frac{df}{ds}\right)_{P_0,u}$ = directional derivative of $f(x,y_0)$ at P_0 in the direction of $\vec{u}$).

$$= \lim_{s \to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0+y_0)}{s}$$

$$\left(\frac{df}{ds}\right)_{P_0,\vec{u}} = \left(D_{\vec{u}} f\right)_{P_0}$$

Ex) $f(x,y) = x^2+xy$. $P_0(1,2)$

$$\vec{u} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j.$$

$$\frac{df}{ds} = \lim_{s \to 0} \frac{(1+\frac{s}{\sqrt{2}}, 2+\frac{s}{\sqrt{2}}) - f(1,2)}{s}$$

$$= \lim_{s \to 0} \frac{(1+\frac{s}{\sqrt{2}})^2 + (1+\frac{s}{\sqrt{2}})(2+s/\sqrt{2}) - 3}{s}$$